Factoring

Techniques for experienced mental calculators

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Factoring

Factoring is breaking down a given number in the smallest possible factors, prime numbers.

The definition of prime number is that such a number is:

- An integer number
- Bigger than one
- Only divisible by itself and by one

The factoring of a number is unique, the order of the factors is indifferent. The number 1 is not considered as a prime number, because than suddenly an endless number of factorings is possible, and the value of the number remains equal. Eg. 15.

15=3×5=1×3×5=1×1×5×3×1×1×1 etc. etc.

Any even number is divisible by 2, and so 2 is the only even prime number and any even number greater than 2 cannot be prime. A number greater than 1 which is not prime is said to be **composite.**

Concerning myself: since 06-01-2011 I am the holder of the world record “Factoring”, which concerns the factoring of five digit numbers, from 10.000 – 99.999, amongst them no prime numbers. The task was to factorize a set of twenty of such numbers. The time needed was measured and because this was the first record attempt the time needed by me – 13 minutes and 38 seconds – this is time is the record .to be broken.

To give an impression

Analysis of the factoring of 5 digit numbers:

From 1-100.000 there are 9.592 prime numbers
From 1-10.000 there are 1.229 prime numbers
Therefore from 10.000 tot 100.000 there are 8.363 prime numbers
From 10.000 - 99.999:
Even numbers: 45.000
Odd numbers: 45.000

Amongst them divisible 36.637
Even numbers: 45.000
To be factorized 81.637 numbers
Amongst them even = 55,12%

And by consequence odd numbers 36.637 = 44,88%.

The biggest prime number we see is 49.999, which we get by factoring the number 99.998. From 2-49.999 there are 5.134 prime numbers

The current working method for factoring is take the square root of the given number and divide the number by all the prime numbers no larger than this root. Eg. √ 100.000=316+, so we need only the 65 prime numbersup to 316 . If a division is exact i.e. there is no remainder, the answer is found. This means a huge quantity of mechanical labor, with the risk of errors. In fact this a rough technique, which generally is defined as “trial and error”.

As tools for factoring I use:

- My knowledge of all the multiplications of the two digit numbers
- My insight in the numbers

- The filtermethods which I elaborate in this article
- My ready knowledge of the prime numbers up to 5.000

My attention was focused to develop filter methods which reduce the mechanical labor, to work faster and to be more accurate. It is also to be realized that the more the numbers are getting bigger, the more time the divisions take.

The number of squares between 10.000and 100.000 is 216 of which I hardly encountered any during my trainings. These can quickly be cast out.

Besides there are the numbers which can quickly be filtered out by mechanical labor: 2,3,5,7,11,13He who commands the 2×2 multiplications can start with this and the primes up to 100 will bring no skilled mental calculator any trouble.

Being about 14 years old I discovered the structure in the squares. Between 1-100 the squares of numbers that do not end in 0 or 5 can be subdivided in groups of four numbers with 4 identical final digits.

Table 1

2^2	4	3^2	9	7^2	49	8^2	64
48^2	2304	47^2	2209	43^2	1849	42^2	1764
52^2	2704	53^2	2809	57^2	3249	58^2	3364
98^2	9604	97^2	9409	93^2	8649	92^2	8464

Immediately is to be seen that the sums of the first and the last number is 100. This is also the case with the second and third number.

Between 1 en 1.000 the squares of numbers not ending in 0 or 5 can be subdivided in groups of eight numbers with three identical final digits, see table 2..

Table 2

11^2	121	23^2	529
239^2	57121	227^2	51529
261^2	68121	273^2	74529
489^2	239121	477^2	227529
511^2	261121	523^2	273529
739^2	546121	727^2	528529
761^2	579121	773^2	597529
989^2	978121	977^2	954529

Here too the symmetry is visible: when adding the first and last numbers, the second and seventh etc. you’’ll see that the sum is always 1.000.

Aside from this I remark that from three digits on there are still more of this kind of groups to be composed.The five digit numbers not ending in 0 or 5 can be divided into groups of eight numbers which have five identical final digits when squared.

Playing with numbers and squares as a 15 year old boy I made a particular discovery. If a number can be written as the sum of two different squares in two different ways, this number can be factorized in two factors which each on itself are the sum of two different squares. What I found also is that if a number can be written only once as the sum of two different squares, it is a prime number. Inb my way of thinking there are two kinds of prime numbers, 3(4) as eg 3,7,11, 19, 23 and the special primes, 1(4) which can be written as one time the sum of two different squares.

Formulated more precisely: if a number is 1(4) and can be expressed as the sum of two different squares in just one way and the number is not divisible by a number which is the square of a number 3(4), then it is a prime number. Eg 41 = 4^2+5^2 which have no common divider and by consequence 41 is a prime number.

Because this article will also be read by mathematicians I – arithmetician – have it read by Dr. B.M.M. de Weger, promoted mathematician and assistant professor at the technical University in Eindhoven. He brought me the disillusioning news that the mathematician Pierre de Fermat has already proved in 1640 that every prime number being 1(4) is the sum of two different squares, but he did not publish the proof. The first published proof comes from Leonard Euler, in 1749.

However: it is completely questionable if these mathematicians hereby have thought on a method for factoring, as in the literature here about nothing is to be found.

Well it is known that every prime number 1(4) can only be written in one way asthe sum of two different squares. And a number 3(4) can per definition not be the sum of two different squares, as every square is either 0(4) or 1(4).

Between 10.000 and 100.000 there are 5.052 odd numbers which can be written as at least twice the sum of two different squares in at least two different ways.Out of them 2.215 are less than 50.000. Eg . 10.001 = 100^2+1^2 and 76^2+65^2. As soon as a composition like this has been found, the factors are quite simple to derive . (100±76)÷2 = 88 and 12. (65±1)÷2= 33 and 32. 88 and 33 have 11 as a common factor, resp. 8× and 3×. 32 and 12 have 4 as a common factor, resp. 8× and 3×. Then the factors of 10.001 are $11^2+4^2=137$ and $8^2+3^2=73$.

Tobe able to execute these operations a profound knowledge of the squares is a necessity.

Over 55 years I lived in the illusion that the above mentioned was strictly new and I looked for a favorable occasion to publish it. And in silence I was very proud of my “invention”.

In relation to the above mentioned I discovered that if one takes the number of special primes being N, then the possibilities for combination are $2^{(n-1)}$. Four of this kind of numbers give 2^3 possibilities for combination eg: : 5×13×17×29= 32045 is tobe composed with the sum of the following pairs of squares : {179+2}, {178+19}, {173+46}, {166+67}, {163+74}, {157+86}, {142+109}, {131+122}. This is only valid when all prime factors are special primes. If one of the special primes is squared there are not four, but three combinations possible and one combination is a multitude of the number which is squared. Eg $325=1^2+18^2=17^2+6^2=15^2+10^2$.

This was also known by Diophantes already in the 3^ecentury .

It is also of paramount importance to know which combinations are possible and which ones not. Considered up to 100 there are 20 rows of squares with different final digits. The numbers ending on 5 have squares ending on 25, and those ending in 0 have squares ending in 00. Per definition squares are either 0(4) or 1(4). Therefore the sums of two squares are either 0, 1 or 2 (4). The table hereunder is a good tool to see which combinations are possible or impossible.

A useful expedient to see how the sum of squares is composed is modulo 9 calculation. If a number offered is 0(9) there is no other possibility than each of the squares is 0(9) . The basic numbers 0/8 squared are resp. 0, 1, 4, 0, 7, 7, 0, 4, 1 en 0(9). If the sum of 2 squares is 1,4 of 7 (9) then automatically one of these squares is 0(9). Numbers 2,5 of 8 (9) are composed of squares 1,4 of 7 (9) in any combination.

This technique I consider as the first filter. With a filter I mean a technique to reduce the theoretically possible number of combinations possible resp. to find the right combination.

As in table 3 we work with hundreds, modulo 16 will be used. Table 3 below considers all possible values for the tens and units, and states whether the corresponding value in the hundreds column is odd or even.

Table 3

	Tens	(H)undreds (E)ven / (O)dd
Final digit 1	01	Always even
	21	Always odd
	41	Always even
	61	Always odd
	81	Always even
Final digit 4	04	E with H0(4) and O with H 3(4)
	24	E with H 2(4) and O with H 3(4)
	44	E with H 6 and 14(16) and O with H 1 and 13 (16)
	64	E with H 0 and 4 (16) and O with H 1 and 9(16)
	84	E with H 4 and 12(16) and O withH 3 and 7 (16)
Final digit	25	Always even, H never 4 of 8
Final digit 6	16	E with H 0 and 12(16), O with H 5 and 13(16)
	36	E with H 0 and 8 (16) and O with H 3 and 15(16)
	56	E with H 2 and 6 (16) and O with H 3 and 11 (16)
	76	E with H 6 and 14 (16) and O with H 1 and 9 (16)
	96	
Final digit 9	09	Always even , H allE(16): 0,2,4,6,8,10,12,14
	29	Always odd, H allO(16) : 1,3,5,7,9,11,13,15
	49	Always evenH all E (16): 0,2,4,6,8,10,12,14
	69	Always odd , H all O (16): 1,3,5,7,9,11,13,15
	89	Always even H, all E (16): 0,2,4,6,8,10,12,14

Examples of sums of squares:

8.641= 1(9). One of the squares has to be 0 (9) and the other one 1(9). 41 is tobe composed by adding xx16(EH) + 25 (always even hundreds) or xx41 (even hundreds) + 00 (even hundreds). 4^2=7(9) so it drops out , 96^2=9.216=> 8.641, drops out too. . 21^2 = 441, 8641-441=8.200 which is no square, 29^2 = 841=4(9), drops out too. 71^2= 5041+ 3600 = 8641. A full hit! There are no other possibilities for combination, so 8641 is prime. For the good order: 79^2 ends indeed in 41 but drops out as it is 4(9) .

6.953. = 5(9). (OH)53 is tobe composed as follows . 09(always EH) + 44(OH), 29 + 24(EH), 64 (EH) + 89 or 84(OH) + 69. The number of possibilities for choice can considerably be reduced by doubling the number. 13906 (1)9 can only be composed by xx81 and xx 25 where one of the squares is 0(9) and the other one is 1 (9). . 9^2=81, but 13.825 cannot be a square. 41^2 and 59^2 are 7(9) so drop out. 91^2=8.281= 1(9). 13.906-8.281=5.625=75^2, so is a hit . Furthermore we find 109^2+45^2. We do now retrospective labor, by dividing by 2. Then (91±75)÷2equals 83 and 8 and (109 ± 45)÷2 equals 77 and 32. Then we do with the odd numbers (83±77)÷2 and then get 80 and 3. The even numbers (32±8)÷2 and then we get 20 and 12. The common factor of 80 and 20 = 20, resp. 4× en 1×. The common factor of 12 and 3 is 3, resp. 4× en 1×. So the factors of 6.953 are 20^2+3^2= 409 and 4^2+1^2=17.

We now have had two filters: modulo 9 and the table 3.

To reduce the quantity of work to be done –GertMittring “a good calculator is lazy”- it is recommended to divide the given number firstly by all the primes < 100.This can be done very quickly because of the small numbers and the rate of errors is minimal. In theory there are 25 divisions out of which the primes 2 to 19 almost automatically drop out.

We start to work with the first prime number >100: 101 . In theory this is the smallest factor, the biggest one is 991 as 101×991= 100.091. Furthermore 101+991= 1092. This is the maximum sum of two prime factors. The maximum square we have to deal with will be the half of it: 546.

The biggest square we have to add with the number offered will not need to complete to more than 546^2=298116. In theory this is 445^2=198025, because 100.091 is 2(9) which number only can be completed to a square with a square 7(9) . The biggest prime factor is 313, after all 317^2=100489. Between 101 and 313 there are 40 primes.

Example **88.183.** √88.183=296,9. The biggest prime number <297=293. The primes up to 100 are used for division. From 101 up to 293 remain 36 prime numbers. One should be aware that the risk of errors is much greater when dividing by larger numbers.

Our first conclusion is that one of the factors of 88183 must lie between 101 and 293. This is selection one.

Because we know that 88183 is not prime, given the problem, we can use the formula $(a^2-b^2)= (a+b)(a-b)$ where 88183 represents (a+b)(a-b), to withwhich number he have to add b^2 to obtain a^2 .

The smallest factor, a-b, is > 101 because we have tried all factors up to 100. If 101 should be the smallest possible factor, 877 will be the biggest one: 101×877=88. 577. This is selection two.

(101+877)÷2=489^2=239121. 239121-88183=± 151.000. 388^2=150544, the square we are looking for then lies between 1 and 388. This third selection brings us this: 88.183=1(9) so that the square we have to add herewith has to be 0(9) and by consequence the basic number of it will be 0(3).

From 0 up to 9 the modulo 9 squares of the successive numbers are: 0, 1, 4, 0, 7, 7, 0, 4, 1, 0. If eg. (a+b)(a-b) =0 and we should find a square to add to another square, b^2 has to be 0(9) . If (a+b)(a-b) =1 the square to be found has to be 0(9).

The fourth selection: the square required has to end in 1and even tens, after all the other final digits of the squares – 0,4,5,6,9- added with the 3 of the number offered would lead to a number which cannot be a square. A square ending in 6 has automatically an odd ten – 16,36,56, 76 or 96. Adding such a square to 88.183 would give a number ending in 9 with an odd ten. Such a number can never be a square, see table 3.

So the following numbers match the criteria : 9, 21, 39, 51, 69 t/m 381, so with jumps of 30. So up to 381 there are 26 candidates.

Now we have found that none of the factors are < 101, the difference between the number offered and 101^2=10201 is ± 78.000 which means that the square tobe found will not lie more than 78.000 above 88.183 so that 279is the highest number we need to try. This is selection 5.

As the sixth selection we now invoke the help of modulo 16, as 16 has no common divisors with 9 and modulo 4 offers to few possibilities for composing squares. Eg 88.183 = 7(16), if we add to that a square 9(16) we have a square with basic number 0(4), with which we can reduce the number of possibilities rather than a number which is 2(4). The squares of the numbers from 1 as far as inclusive 15 are respectively 1, 4, 9, 0, 9, 4, 1, 0, 1, 4, 9, 0, 9, 4, 1.

Table 4 gives us the mod 9 and mod 16 properties of the squares up to 72. Every number mentioned in table 4 can be increased by 72 until the completion wanted is obtained.

Table 4

Moduli of the squares:
Mod 9: 0, 1, 4, 0 , 7, 7, 0, 4, 1.

Mod 16: 0, 1, 4, 9, 0, 9, 4, 1, 0, 1, 4, 9, 0, 9, 4, 1.

Rule nr	Mod. 9 resp. 16	Basic number of squares possible					
1	0-0	12	24	36	48	60	
2	1-0	8	28	44	64		
3	4-0	16	20	52	56		
4	7-0	4	32	40	68		
5	1-1	1	17	55	71		
6	1-4	10	26	46	62		
7	1-9	19	35	37	53		
8	4-0	16	20	52	56		
9	4-1	7	25	47	65		
10	4-4	2	34	38	70		
11	4-9	11	29	43	61		
12	7-0	4	32	40	68		
13	7-1	23	31	41	49		
14	7-4	14	22	50	58		
15	7-9	5	13	59	67		
16	0-1	9	15	33	39	57	63
17	0-4	6	18	30	42	54	66
18	0-9	3	21	27	45	51	69

88.183= 1(9) and 7(16) which includes that b² has to be 0(9) and 9(16) according to table 10.

To have a good insight in the squares tobe added follows now table 5 .

Table 5

Tens		To complete to		To complete to	
Even	Units	With	To	With	to
	1	00	01	04, 24, 44, 64, 84	25
	3	01, 21, 41, 61, 81	04, 24, 44, 64, 84		
	7	09, 29, 49, 69, 89	16, 36, 56, 76, 96		
	9	00	09 ,29, 49, 69, 89	16, 36, 56, 76, 96	25
Odd	1	25	16, 36, 56, 76, 96	09, 29, 49, 69, 89	00
	3	16, 36, 56, 76, 96	09, 29, 49, 69, 89		
	7	04, 24, 44, 64, 84	01, 21, 41, 61, 81		
	9	25	04, 24, 44, 64, 84	01, 21, 41, 61, 81	00

The seventh selection is the squares 1(9) and 0(16), the basic numbers therefore 1 or 8(9) and 0(4), <88183 and not bigger than ± 166.000. The possibilities are 332,352,368,388. We start with 332^2=110.224-88.183= 22.041. By this the numbers 51, 69, 99 and 141 drop of, as their square < 22.000.

So we can start with + 171^2=29.241+88.183=117424, which is no square.. + 189^2=35.721+88.183=123.904, this is a "full" square, of352^2. We now have found the solution. 88.183= (352+189) × (352-189) = 541×163.

Another example: a number ending on xx17 can be completed with a number yy04 to a number zz21: 23.617 + 48^2= 25.921=161^2. Or xx17 +yy44 = zz61: 44.117 + 3.844(62^2)= 47.961= 219^2. Table 6 gives some more concrete applications.

Table 6

Examples	+	Gives	= ²	+	Gives	
2.101	8.100	10.201	101^2			
6.901	324	7.225	85^2			
943	81	1.024	32^2			
4.307	49	4.356	66^2			
9.709	900	10.609	103^2			
21.409	9.216	30.625	175^2			
3.111	25	3.136	56^2	6.889	10.000	100^2
3.713	256	3.969	63^3			
7.117	101.124	108.241	329^2			
22.019	4.225	26.244	162^2			
60.279	1.225	61.504	248^2	151.321	211.600	460^2

For enlightenmentanotherexample:

24.779=2(9) and 11(16). The square tobe added must be 7(9) and 9(16) and can be xx21, together yy00(OH) or xx25, together yy04(EH)). Xx21 candidates have tobe with the hundreds 1, 3 or 7 as the hundreds 5 and 9 lead to hundreds which never can be a square: 300 or 700. Xx21 Candidates 139, 239, 311. xx25 Candidates 05,85, 275, 355, 365. √24.779=157+, so the smallest factor lies between 7 and 157. By quick divisions we can cast out the numbers up to 50. If 53 were the smallest factor, its counterpart should be ±460, which in principle is possible. We now can select the candidates which meet the requirements 7(9) and 9(16) and lie between 53 and 157.

Concerning xx25 we have 05,85, 275, 355, 365, which squared and added with 24.779 give: 24.804, 32.004, 158.004, 222.804, none of them being squares. Concerning xx21 gives 139^2 immediately a full hit as 24.779+19.321= 44.100=210^2. So the factors of 24.779 are 210±139, 71 en 349.

Very shortly I discovered another technique, on which I am very proud.

Thinking on and playing with numbers to my big joy and surprise I discovered the following. If a number cannot be written as twice the sum of two different squares at all and it is 1(4) then this number can be factorized into two numbers each of them being 3(4). My theoretical knowledge is insufficient to give an explanation for this. In this relation it is pleasant to remark that Robert Fountain, a promoted nuclear physicist, described my activities with the nice word "number practitioner". Because I do not like to cheat myself, I tested my "invention" a big number of times. Because in every case the rightness of my operation was proved, I keep it for certain.

But also here a disillusioning surprise awaited me: Dr. De Weger let me know that also this subject in the past was discovered by the mathematicians De Fermat and Euler. In the literature however there is no evidence that these mathematicians hereby have thought on a technique for factorization.

I give some more examples:

301= 7×43, 437= 19×23, 22.493= 83×271,95.477= 307×311.

Finally I answer on a possibly raised question: “ Why must the factorization world record concern five digit numbers?”. Although they are not acknowledged as these by the mental calculators, there exist “memory acrobats”. These are persons with an enormous capacity of memory, there is thought on more than 400.000 numbers, amongst which with up to 200 digits per number. Out of this enormous capacity answers are reproduced and if necessary a bit of calculation. The “average” audience cannot discriminate the difference, a lightning speed and correct answer is presented and this is enough. As there is the possibility that such a memory expert learns by heart the factorization of all four digit numbers the value of a record would seriously be reduced. Learning by heart the factorization of all the five digit numbers is deemed to be impossible.

In 2006 thanks to my knowledge and experience with the prime numbers – I was the only mental calculator who scored in a tournament the full 100 points, and others came to only 25, 12,5 and even 0,00 – Robert Fountain, mentioned before, gave me the very nice epithetonornans “William Flash, King of the Primes”. As far as I know on this moment the American Jerry Newport has also a great knowledge of and experience with the prime numbers and factorization.

The last 3 pages of this article contain tables beginning with 0 mod 9 up to 8 mod 9 in combination with mod 16. They help you to find a completing square for the number you are working with. Eg. you have a number 2(9) and 7 (16) eg 119. In table 11 you’ll find suggestions for the square to complete to another square you are looking for. For 119 you can take 5^2 to complete to 12^2. 407 has the same properties. You can complete this number with 13^2 to 24^2.

Alphen aan den Rijn, 28-08-2012,

Willem Bouman

Table 9				To complete with						
Mod 9		Mod 16		P.E.	+(9)		(16)		Possible	P.E.
0	(9)	0	(16)	144	1,4,7	(9)	1,4,9	(16)		$144+35^2=37^2$
0	(9)	1	(16)	81	1,4,7	(9)	0	(16)	12,4	$81+12^2=15^2$
0	(9)	2	(16)	18		(9)		(16)		
0	(9)	3	(16)	99	1,4,7	(9)	1	(16)	1,15,49	$99+49^2=50^2$
0	(9)	4	(16)	180	1,4,7	(9)	0	(16)	8,44	$180+44^2=46^2$
0	(9)	5	(16)	117	0	(9)	4	(16)	2,18	$117+2^2=11^2$
0	(9)	6	(16)	54		(9)		(16)		
0	(9)	7	(16)	135	0	(9)	9	(16)	all 0(3)	$135+3^2=12^2$
0	(9)	8	(16)	72	0	(9)		(16)	9,15,33,39,57,63	$72+3^2=9^2$
0	(9)	9	(16)	153	0	(9)	1,4,9	(16)	4,24,76	$153+4^2=13^2$
0	(9)	10	(16)	90		(9)		(16)		
0	(9)	11	(16)	27	1,4,7	(9)	1,4,9	(16)	3,13	$27+13^2=14^2$
0	(9)	12	(16)	108	1,4,7	(9)	4	(16)		$28+6^2=8^2$
0	(9)	13	(16)	45	1,4,7	(9)	4	(16)	6,22	$45+6^2=9^2$
0	(9)	14	(16)	126		(9)		(16)		
0	(9)	15	(16)	63	1,4,7	(9)	1	(16)	9,31	$415+39^2=44^2$

Table 10				To complete with						
Mod 9		Mod 16		P.E..	+(9)		(16)		Possible	P.E.
1	(9)	0	(16)	64	0	(9)	all	(16)	all 0(3)	$64+6^2=10^2$
1	(9)	1	(16)	1	0	(9)	0	(16)	6,12,18,24,36,48	$145+12^2=17^2$
1	(9)	2	(16)	82	0	(9)		(16)		
1	(9)	3	(16)	19	0	(9)	1	(16)	9,15,33,39,57,63	$451+15^2=26^2$
1	(9)	4	(16)	100	0	(9)	0	(16)	12,24,36,48,60	$100+24^2=26^2$
1	(9)	5	(16)	37	0	(9)	4	(16)	6,18,30,42,54,66	$325+6^2=19^2$
1	(9)	6	(16)	118	0	(9)		(16)		
1	(9)	7	(16)	55	0	(9)	9	(16)	all 0(3)	$775+3^2=28^2$
1	(9)	8	(16)	136	0	(9)		(16)	9,15,33,39,57,63	$136+33^2=35^2$
1	(9)	9	(16)	73	0	(9)	1,4,9	(16)	12,24,36,48,60	$217+12^2=19^2$
1	(9)	10	(16)	10	0	(9)		(16)		
1	(9)	11	(16)	91	0	(9)	1,4,9	(16)	0	$91+3^2=10^2$
1	(9)	12	(16)	28	0	(9)	4	(16)	6,12,18,24,36,48	$28+6^2=8^2$
1	(9)	13	(16)	109	0	(9)	4	(16)	6,12,18,24,36,48	$253+6^2=17^2$
1	(9)	14	(16)	46	0	(9)		(16)		
1	(9)	15	(16)	127	0	(9)	1	(16)	9,15,33,39,57,63	$415+39^2=44^2$

Table 11				To complete with						
Mod 9		Mod 16		P.E.	+(9)		(16)		Possible	P.E.
2	(9)	0	(16)	128	7	(9)	0	(16)	4,32,40,68	$128+14^2=18^2$
2	(9)	1	(16)	65	7	(9)	0	(16)	4,32,40,68	$65+4^2=9^2$
2	(9)	2	(16)	2	7	(9)		(16)		
2	(9)	3	(16)	83	7	(9)	1	(16)	23,31,41,49	$371+23^2=30^2$
2	(9)	4	(16)	20	7	(9)	0	(16)	4,32,40,68	$20+4^2=6^2$
2	(9)	5	(16)	101	7	(9)	4	(16)	14,22,50,58	$101+50^2=51^2$
2	(9)	6	(16)	38	7	(9)		(16)		
2	(9)	7	(16)	119	7	(9)	9	(16)	5,13,59,67	$119+5^2=12^2$
2	(9)	8	(16)	56	7	(9)	1	(16)	5,13,59,67	$56+52=9^2$
2	(9)	9	(16)	137	7	(9)	0	(16)	4,32,40,68	$425+4^2=21^2$
2	(9)	10	(16)	74	7	(9)		(16)		
2	(9)	11	(16)	11	7	(9)	9	(16)	5,13,59,67	$155+13=18^2$
2	(9)	12	(16)	12	7	(9)	4	(16)		
2	(9)	13	(16)	13	7	(9)	4	(16)	14,22,50,58	$29+14^2=15^2$
2	(9)	14	(16)	14	7	(9)		(16)		
2	(9)	15	(16)	15	7	(9)	1	(16)	23,31,41,49	$335+31^2=36^2$

Table 12					To complete with					
Mod 9		Mod 16		P.E.	+(9)		(16)		Possible	P.E.
3	(9)	0	(16)	192	1,4,7	(9)	0,4	(16)	2,8,22	$192+22^2=26^2$
3	(9)	1	(16)	273	1,4,7	(9)	0,4	(16)	4,16	$273+16^2=23^2$
3	(9)	2	(16)	66		(9)		(16)		
3	(9)	3	(16)	147	1,4,7	(9)	1	(16)	1,7,23	$147+23^2=26^2$
3	(9)	4	(16)	84	1,4,7	(9)	0	(16)	4, 20	$84+20^2=22^2$
3	(9)	5	(16)	165	1,4,7	(9)	4	(16)	2, 10	$165+26^2=29^2$
3	(9)	6	(16)	102		(9)		(16)		
3	(9)	7	(16)	39	1,4,7	(9)	9	(16)	5,19	$39+5^2=8^2$
3	(9)	8	(16)	1209	1,4,7	(9)	9	(16)	13,29	$120+13^2=17^2$
3	(9)	9	(16)	57	1,4,7	(9)	0	(16)	8,28	$57+28^2=29^2$
3	(9)	10	(16)	138		(9)		(16)		
3	(9)	11	(16)	75	1,4,7	(9)	9	(16)	5,37	$75+37^2=38^2$
3	(9)	12	(16)	156	1,4,7	(9)	4	(16)	10	$156+10^2=16^2$
3	(9)	13	(16)	93	1,4,7	(9)	4	(16)	14, 46	$93+14^2=17^2$
3	(9)	14	(16)	30		(9)		(16)		
3	(9)	15	(16)	111	1,4,7	(9)	1	(16)	17,55	$111+17^2=20^2$

Table 13					To complete with					
Mod 9		Mod 16		P.E.	+(9)		(16)		Possible	P.E.
4	(9)	0	(16)	112	0	(9)	0	(16)	12,24,36,48,60	$112+12^2=16^2$
4	(9)	1	(16)	49	0	(9)	0	(16)	12,24,36,48,60	$481+12^2=25^2$
4	(9)	2	(16)	130		(9)		(16)		
4	(9)	3	(16)	67	0	(9)	1	(16)	9,15,33,39,57,63	$355+33^2=38^2$
4	(9)	4	(16)	4	0	(9)	0	(16)	12,24,36,48,60	$148+36^2=38^2$
4	(9)	5	(16)	85	0	(9)	4	(16)	6,18,30,42,54,66	$805+78^2=83^2$
4	(9)	6	(16)	22		(9)		(16)		
4	(9)	7	(16)	103	0	(9)	9	(16)	3,21,27,45,51,69	$247+3^2=16^2$
4	(9)	8	(16)	40	0	(9)	1	(16)	9,15,33,39,57,63	$184+21^2=25^2$
4	(9)	9	(16)	121	0	(9)	0	(16)	12,24,36,48,60	$265+24^2=29^2$
4	(9)	10	(16)	58		(9)		(16)		
4	(9)	11	(16)	139	0	(9)	9	(16)	3,21,27,45,51,69	$427+27^2=34^2$
4	(9)	12	(16)	76	0	(9)	4	(16)	6,18,30,42,54,66	$76+18^2=20^2$
4	(9)	13	(16)	13	0	(9)	4	(16)	9,15,33,39,57,63	$3-1+18^2=25^2$
4	(9)	14	(16)	94		(9)		(16)		
4	(9)	15	(16)	31	0	(9)	1	(16)	9,15,33,39,57,63	$319+9^2=20^2$

Table 14					To complete with					
Mod 9		Mod 16		P.E.	+(9)		(16)		Possible	P.E.
5	(9)	0	(16)	32	4	(9)	4	(16)	2,34,38,70	$608+34^2=42^2$
5	(9)	1	(16)	113	4	(9)	0	(16)	16,20,52,56	$545+52^2=57^2$
5	(9)	2	(16)	50		(9)		(16)		
5	(9)	3	(16)	131	4	(9)	1	(16)	7,25,47,65	$275+25^2=30^2$
5	(9)	4	(16)	68	4	(9)	0	(16)	16,20,52,56	$68+16^2=18^2$
5	(9)	5	(16)	5	4	(9)	4	(16)	2,34,38,70	$437+2^2=21^2$
5	(9)	6	(16)	86		(9)		(16)		
5	(9)	7	(16)	23	4	(9)	9	(16)	11,29,43,61	$203+11^2=18^2$
5	(9)	8	(16)	104	4	(9)	1	(16)	7,25,47,65	$140+34^2=36^2$
5	(9)	9	(16)	41	4	(9)	0	(16)	16,20,52,56	$185+16^2=$21¡
5	(9)	10	(16)	122		(9)		(16)		
5	(9)	11	(16)	59	4	(9)	9	(16)	11,29,43,61	$203+11^2=18^2$
5	(9)	12	(16)	140	4	(9)	4	(16)	2,34,38,70	$284+70^2=72^2$
5	(9)	13	(16)	77	4	(9)	4	(16)	2,34,38,70	$77+38^2=39^2$
5	(9)	14	(16)	14		(9)		(16)		
5	(9)	15	(16)	95	4	(9)	1	(16)	7,25,47,65	$527+7^2=24^2$

Table 15

Mod 9		Mod 16		To complete with P.E.	+(9)		(16)		Possible	P.E.
6	(9)	0	(16)	96	1,4,7	(9)	4	(16)	2,10,23	$96+23^2=25^2$
6	(9)	1	(16)	177	1,4,7	(9)	0	(16)	16,28	$177+28^2=31^2$
6	(9)	2	(16)	114		(9)		(16)		
6	(9)	3	(16)	51	1,4,7	(9)	1	(16)	7,25	$51+25^2=26^2$
6	(9)	4	(16)	132	1,4,7	(9)	0	(16)	4,12	$132+32^2=34^2$
6	(9)	5	(16)	69	0	(9)	4	(16)	10,34	$69+34^2=35^2$
6	(9)	6	(16)	6		(9)		(16)		
6	(9)	7	(16)	87	0	(9)	0	(16)	13,43	$87+13^2=16^2$
6	(9)	8	(16)	168	0	(9)	1,9	(16)	1,11,41	$168+1^2=13^2$
6	(9)	9	(16)	105	0	(9)	0	(16)	8,52	$105+8^2=132$
6	(9)	10	(16)	186		(9)		(16)		
6	(9)	11	(16)	123	1,4,7	(9)	9	(16)	19,61	$123+61`^2=62^2$
6	(9)	12	(16)	60	1,4,7	(9)	4	(16)	14, 50	$60+14^2=16^2$
6	(9)	13	(16)	141	1,4,7	(9)	4	(16)	7,73	$147+73^2=74^2$
6	(9)	14	(16)	78		(9)		(16)		
6	(9)	15	(16)	159	1,4,7	(9)	1	(16)	7,25	$159+79^2=80^2$

Table 16

Mod 9		Mod 16		To complete with P.E.	+(9)		(16)		Possible	P.E.
7	(9)	0	(16)	16	0	(9)	0	(16)	12,24,36,48,60	$304+36^2=40^2$
7	(9)	1	(16)	97	0	(9)	0	(16)	12,24,36,48,60	$385+12^2=23^2$
7	(9)	2	(16)	34	0	(9)		(16)		
7	(9)	3	(16)	115	0	(9)	1	(16)	9,15,33,39,57,63	$259+15^2=22^2$
7	(9)	4	(16)	52	0	(9)		(16)		
7	(9)	5	(16)	133	0	(9)	4	(16)	6,18,30,42,54,66	$565+54^2=59^2$
7	(9)	6	(16)	70	0	(9)		(16)		
7	(9)	7	(16)	7	0	(9)	9	(16)	3,21,27,45,51,69	$295+27^2=32^2$
7	(9)	8	(16)	88	0	(9)		(16)		
7	(9)	9	(16)	25	0	(9)	0	(16)	12,24,36,48,60	$25+12^2=13^2$
7	(9)	10	(16)	106	0	(9)		(16)		
7	(9)	11	(16)	43	0	(9)	9	(16)	3,21,27,45,51,69	$475+3^2=22^2$
7	(9)	12	(16)	124	0	(9)		(16)		
7	(9)	13	(16)	61	0	(9)	4	(16)	6,18,30,42,54,66	$205+18^2=23^2$
7	(9)	14	(16)	142	0	(9)		(16)		
7	(9)	15	(16)	79	0	(9)	1	(16)	9,15,33,39,57,63	$511+33^2=40^2$

Table 17

Mod 9		Mod 16		To complete with P.E..	+(9)		(16)		Possible	P.E.
8	(9)	0	(16)	80	1	(9)	1	(16)	8,28,44,64	$80+8^2=12^2$
8	(9)	1	(16)	17	1	(9)		(16)	8,28,44,64	$161+8^2-=15^2$
8	(9)	2	(16)	98	1	(9)		(16)		
8	(9)	3	(16)	35	1	(9)	1	(16)	1,17,55,71	$611+17^2=30^2$
8	(9)	4	(16)	116	1	(9)		(16)		
8	(9)	5	(16)	53	1	(9)	4	(16)	10,26,46,62	$341+10^2=21^2$
8	(9)	6	(16)	134	1	(9)		(16)		
8	(9)	7	(16)	71	1	(9)	9	(16)	19,35,37,53	$215+19^2=24^2$
8	(9)	8	(16)	8	1	(9)		(16)		
8	(9)	9	(16)	89	1	(9)		(16)	8,28,44,64	$377+8^2=21^2$
8	(9)	10	(16)	26	1	(9)		(16)		
8	(9)	11	(16)	107	1	(9)	9	(16)	19,35,37,53	$395+37^2=42^2$
8	(9)	12	(16)	44	1	(9)		(16)		
8	(9)	13	(16)	125	1	(9)	4	(16)	10,26,46,62	$413+26^2=33^2$
8	(9)	14	(16)	62	1	(9)		(16)		
8	(9)	15	(16)	143	1	(9)	1	(16)	1,17,55,71	$1007+17^2=36^2$