Diophantine Equations

This chapter concerns a special type of Diophantine equations. The name comes from the Hellenistic mathematician Diophantus from Alexandria, who lived ± 250 A.D. who in his studies of these equations as one of the first introduced symbols in the algebra.

Definition: a Diophantine equation is an indefinite polynomial equation of which the variables may only be integer numbers. Diophantine problems have less equations than unknown variables.

In this chapter we will only speak about equations of the following shape e.g.: $a + b + c = 23$; $a^3 + b^3 + c^3 = 2135$. In this case the solution is $a = 12$; $b = 7$ and $c = 4$. We will describe in which way the solution for this type of questions can be found on an arithmetically .Until now this type of questions did not appear in tournaments for mental calculation. However, in the book “the great mental calculators” we can find an example on page 248. The question is: “Find three numbers such that their sum is 43 and the sum of their cubes is 17.299. The numbers are 25, 11 and 7.

The way of solving this question is not mentioned. This fact triggered me to make a study of this type of questions and after a lot of thinking I found a solution, in fact without any mathematic form, but by sieving out incorrect solutions that long, until a correct one was found. Therefore the subtitle of this chapter: the Sieve of Willem Bouman. The clue, after a lot of thinking was, you will not be surprised: Modulo calculation, in this case mod 27.

For the good order: for solving questions like this by heart a perfect knowledge of the cubes up to 100 is indispensable. For finding the solutions with bigger numbers I used paper to keep a clear overview, but in no way a calculator machine or other expedients.

And again: it is difficult to describe the intense joy I felt, after it appeared that my method works. And even if it takes a lot of "sievings" there is the joy of "yes, again the solution is found".

Well willing friends became instructions: "Please give me ± 20 numbers according to this question $a + b + c = x$ and $a^3 + b^3 + c^3 = z$ and please limit the sum of $a + b + c$ to ± 100.

Later on I was given the instruction for Mathematica as follows:

```
For [i=1, i=<100, i++
a=RandomInteger[{10,99}]
b=RandomInteger[{10,99}]
c=RandomInteger[{10,99}]
x = a+b+c
y = a³ + b³ + c³
Print[{x,y}]]
```

You read here that the instruction consists of:

- Integer numbers, randomly chosen
- Numbers of two figures: 10 – 99
- There are three of them, a, b and c and they are added, but a, b and c themselves are not mentioned!
- A, b and c are calculated to the third power and added: $y = a^3 + b^3 + c^3$;
- The result is printed

This gives one hundred question numbers to be solved. Theoretically the biggest sum is $97^3 + 98^3 + 99^3 = 2824164$, this number was not yet presented to me. The biggest until now is

2186116, the sum of $99^3 + 98^3 + 65^3$.

Before diving in the rough practice it is useful to mention something about modulo calculation in the third power. If we work with two numbers in the odd powers the following happens, if we have for example $3^3 + 4^3$, we calculate 3 + 4 =7. Then we know that the result of the addition $3^3 + 4^3$ will be divisible by 7: $3^3 + 4^3 = 91$ is divisible by 7. In the same way: $3^5 + 4^5 = 1267$, which is divisible by 7, finally $3^7 + 4^7 = 18571$, which also is divisible by 7.

But this principle does not work when we combine three numbers. $3^3 + 19^3 = 6886$, which is divisible by 22. $4^3 + 19^3 = 6923$, which is divisible by 23 = 19 + 4. Let's now take $3^3 + 4^3 + 19^3 = 6950$, which is not divisible by 26 = 19 + 4 + 3.

So if we take the sum of three integer numbers, we cannot deduct something to find any of the three numbers quickly, we have to do it another way.

However, if we know one of the numbers a, b and c, we can do something. Suppose the question is a + b + c = 26 and the sum of their cubes is 6950. After having subtracted 3^3 and have 6923 we may conclude that one of the factors of 6923 will be 19 + 4 = 23. In the same way if a = 4 and we subtract 4^3 and have 6886 and then conclude that one of the factors will be 22.

Now I am going to reveal my method, I hope you can share with me the deep satisfaction I felt after having seen it all works. Before I had the Mathematica instruction some people were so kind to make questions for me, because as you know there is one "holy" principle: one should never make his own questions. At a given moment I could not find the answer and spent a lot of paper with calculations. As the questions were made by an acknowledged mathematician I doubted on myself rather than *on him. Ultimately I gathered all my courage and asked very prudently “Is it possible that there is a mistake in question 14, as I cannot find the solution”. Reaction “Excuse me, I made a mistake in the addition of the cubes, this is the correct number”. And then, within a minute finding the solution, yes this gives a very rich satisfaction…..*

Mod 9 cubed	Gets mod 27	Mod 9 cubed	Gets mod 27
1	1	5	17
2	8	6	0
3	0	7	19
4	10	8	26

Ok, we work modulo 27, according to the table hereunder.

To be prepared for surprises a lot of possibilities will be worked out.

Before starting with mod 27 it is good to study the “conversion table mod 0 / mod 27”.

Table 1

Table 2

X	X^3	Mod 27	X	X^3	Mod 27	X	X^3	Mod 27
1	1	1	10	1000	1	19	6859	1

2	8	8	11	1331	8	20	8000	8
3	27	0	12	1728	0	21	9261	0
4	64	10	13	2197	10	22	10648	10
5	125	17	14	2744	17	23	12167	17
6	216	0	15	3375	0	24	13824	0
7	343	19	16	4096	19	25	15625	19
8	512	26	17	4913	26	26	17576	26
9	729	0	18	5832	0	27	19683	0

Studying this table we find:

- 9 times 0
- The moduli 1, 8, 10, 17, 19, and 26 appear each of them 3 times
- 1 + 26 = 0; 8 + 19 = 0; 10 + 17 = 0, of course all mod 27.
- What remains are the moduli 1, 8, 10, 17, 19 and 26

But do not forget: Our problems concern the sum of three of these moduli. Based on this 4, 5, 13, 14, 22, 23 as sums are impossible. Anything else is possible.
So, we have 21 possibilities, and 6 impossible modulo 27 outcomes. In a table the possibilities are to be seen.

To make the solutions less try and error a table was made:

In the column “Sum 27” the possibilities are mentioned for the question numbers of the shape $a^3 + b^3 + c^3$ and their mod 27. The three columns beside, (27) enlist the possibilities of combination to come to the sum. E.g. : if the question number is 0(27) this can be obtained by the combinations as mentioned. The column “Common” indicates if there is a common mod 27 in the sum of moduli. If so, this is the best way to start with the common mod 27.

Table 3

Sum (27)	(27)	(27)	(27)	(27)	(27)	Com-mon
0	0,0,0,	0, 8, 19	0.10, 17	0, 1, 26		0
1	0, 0, 1	1, 10, 17	1, 8, 19	1, 1, 26	17, 19, 19	
2	0, 1, 1	0, 10, 19				
3	1, 1 ,1	10, 10, 10	19, 19,19	1, 10, 19		
6	17, 17, 26	8, 26, 26	8, 8, 17			
7	0, 8, 26	0, 17, 17				0
8	0, 0, 8	8, 10, 17	8, 8, 19	1, 8, 26		8
9	0, 1, 8	0, 17, 19	0, 10,26			0
10	0, 0, 10	1,,1, 8	8,10,19	10, 10 27		
11	0, 19, 19	0, 1, 10				0
12	1, 1, 10	1, 10, 19				1
15	8, 8, 26	17, 26, 26				26
16	0, 8, 8	0, 17, 26				0
17	0, 0, 17	1, 17, 26,	8, 17, 19	10, 17, 17		17
18	0, 19, 26	0, 8, 10	0, 1, 17			0
19	0, 0, 19	8, 19, 19	10, 17, 19	1, 26, 19	1, 1, 17	
20	0, 1, 19	0, 10, 10				0
21	1, 1, 19	10, 19, 19				19
24	8, 8, 8	17, 17, 17	26, 26, 26			
25	0, 8, 17	0, 26, 26				0
26	0, 0, 26	8, 19, 26	10, 17, 26	8, 8, 10	1, 26, 26	

As the practice is the best instructor we start with working out step by step some of the numbers offered to me.

During the 2015 “Rekenwonderweekend” one theme of discussion was: “Can there be found a formula with which we can find without trying one of the three numbers we are so desperately looking for?”, and the answer is NO.

Apart from the indispensable modulo calculation we can see rapidly two things:

- The rounded cube root of the question number destines the maximum a
- Divide the QN by 3 and calculate the rounded cube root. This is the minimum a, and between those two numbers lies one of the numbers we look for

$a^3 + b^3 + c^3 = 684$

Problem: $a^3 + b^3 + c^3 = 684$ and $a + b + c = 18$

The rounded cube root of 684 is 8+, so $a_{max} = 8$. $A_{min} = 684 \div 3 = 228$, the cube root of it is 6+. So one of our candidates lies between 6 and 8+, be it either 6, 7 or 8.

As $a^3 + b^3 + c^3 = 684$, which is 9 (mod 27) the first search is how can we obtain 9 (mod 27) by adding 3 cubes according to table 3? These are the possibilities:

- 0 + 1 + 8
- 0 + 10 + 26
- 0 + 17 + 19

We see 0 is the common number, we now know that the cube of one of the numbers a, b, and c must be divisible by 27. There are two possibilities 3^3 and 6^3, it is unknown which one is the right one.

A try: a = 3, then we have $3^3 = 27 = 0 \pmod{27}$ and $684 – 27 = 657 = 9 \pmod{27}$ which represents $b^3 + c^3$. Now are confronted with a new question: Which two numbers add up 18 - 3 = 15 while the sum of their cubes is 657? Next try:

- $8^3 + 7^3$ = too big 855 and over more 18 (mod 27), the smallest one, in fact other tries do not make sense

- $6^3 + 9^3$ makes no sense, as 9^3 already exceeds 657
- $10^3 + 5^3$ is even worse, 1125 and over more 18 (mod 27)

As the biggest number is “leading” it makes no sense to try $11^3 + 4^3$, we can stop here.

There is another possibility: a = 6, the cube is 216. Next we subtract 684 – 216 = 468 ≡ 9 (mod 27). These are the possibilities:

- $6^3 + 6^3 = 216 + 216 = 432$, this is too few, over more it is conflicting with the idea of three different numbers. And it is 0 (mod 27)
- $7^3 + 5^3 = 468 \equiv 9$ (mod 27). Here we have the solution

So $684 = 5^3 + 6^3 + 7^3$.

This is in principle the way of working. There is no formula, it is all a matter of sieving out the possibilities until the moment of the solution. You will see: the bigger the numbers, the more interesting and challenging it gets!!!

$a^3 + b^3 + c^3 = 2403$

Problem : $a^3 + b^3 + c^3 = 2403$ and $a + b + c = 27$

The rounded cube root of 2403 = 13 +, This is the maximum value of a.

First step: $a^3 + b^3 + c^3 = 0$ (mod 27). A look at table 2 shows that we have these possibilities:

- 0 + 0 + 0
- 0 + 10 + 27
- 0 + 8 + 19
- 0 + 1+ 26

We could be smart and do $27 \div 3 = 9$ and try $a = b = c = 9$. But that cock won't fight: $3 \times 9^3 = 2187$, an important shortfall. One thing is for sure: 0 is the common number in the possibilities, so we firstly have to look after the numbers which's cubes are 0 (mod 27), We have 3, 6, 9 and 12. 15 drops out as 15^3 exceeds the question number by far.

Our first try: $a = 3$.

Then $a^3 + b^3 + c^3 - a^3 = 2403 – 3^3 = 2376$, and $a + b + c – a = 27 - 3 = 24$.

This means that we have to look for two numbers which add up to 24 and of which the sum of their cubes is 2376. We do not need to calculate, even 2×12^3 exceeds 2403 by far.

At this moment it is good to think about the "domination of the biggest number". We take now 13 and 11, their sum is also 24. $13^3 + 11^3 = 2197 + 1331 = 3548$, which exceeds the sum of $12^3 + 12^2$. In the same way $14^3 + 10^3 = 3744$. Further additions are not useful, we can stop here.
Conclusion; $a = 3$ will not lead to the solution of the question.

Next one $a = 6$.

$6^3 = 216 = 0$ (mod 27), of course; now $b^3 + c^3 = a^3 + b^3 +c^3 - a^3$ and $b + c = a + b + c - a = 21$. Can we find two numbers which add up to 21 and for which the sum of their cubes is 2187? The smallest combination is $10^3 + 11^3 = 2331$, so this drops out and by consequence all the other possibilities. The "leading" number wins, $12^3 + 9^3 = 2457$, so we are ready with 6.

Next one $a= 9$.

For $a = 9$ we have $9^3 = 729$, $b^3 + c^3 = a^3 + b^3 + c^3 - a^3$. $2403 – 729 = 1674$. $B + c = a + b + c - a = 18$. According to the possibilities we could try 12 + 6, but already 12^3 exceeds 1674, so it drops out. Then we try $11^2 + 7^3$ and here we are: 1331 + 343= 1674 and 11 + 7 =18. So now we have the final solution is $2403 = 9^3 + 11^3 + 7^3$ and 9 + 11 + 7 = 27.

$a^3 + b^3 + c^3 = 8504$

Problem $a^3 + b^3 + c^3 = 8504 = 26 \pmod{27}$ and $a + b + c = 38$.

The rounded cube root of the question number is 20+, so this is the maximum value of a. The minimum value of a = cube root of 8504 ÷ 3 = 2834, so 14.

As usual, a look at table 3 shows us that the remainder of 26 (mod 27) can be obtained by these additions:

- 0 + 0 + 26
- 1 + 26 + 26
- 8 + 19 + 26
- 10 + 19 + 26

Based on these possibilities it lies at hand that our first search is for a number which is less than 20 and which cube is Ξ26 (mod 27). There are $8^3 = 512$ Ξ 26 (mod 27) and $17^3 = 4913 \equiv 26$ (mod 27)

First possibility a = 8 and $8^3 = 512$. Then $a^3 + b^3 + c^3 - a^3 = b^3 + c^3$. 8504 - 512 = 7992 and a + b + c - a = b + c 30. Next question are there two numbers which's sum is 30 and who's cubes added result in 7992? The lowest possibility is $2 \times 15^3 = 6750$.
We can count up with $2 \times 15^3 = 6750$; $16^3 + 14^3 = 6840$; $17^3 + 13^3 = 7110$; $18^3 + 12^3 = 7560$; too few $19^3 + 11^3 = 8190$, too much.
But we can also use our brains, as follows: the sum of the basic

numbers such as 15 + 15 and 16 + 14, et cetera is 30, which ends on a zero and so do their cubes too, as you can see here above.
So this solution is a dead end.

Second possibility We go on with $a = 17$ and $17^3 = 4913$;

$a^3 + b^3 + c^3 - a^3 = b^3 + c^3$. 8504 – 4913 = 3591 and now $b + c = a + b + c - a = 38 - 17 = 21$. Next question are there two numbers which's sum is 21 and who's cubes added result in 3591.

It can be any of these combinations : 11 + 10; 12 + 9; 13 + 8; 14 + 7; 15 + 6. Here we can stop as 16^3 exceeds 3591. It is not so very difficult to estimate which is the best "guess": $14^3 + 7^3 = 9 \times 343 = 3087$ and indeed, taking $15^3 + 6^3$ we get the 3591 we are looking for. So the solution is $17^3 + 15^3 + 6^3 = 8504$.

There is also another way of working: we have 3591 and the sum 15 + 6 is 21. At the moment a situation like this appears, that is to say that the sum of the cubes is divisible by the sum of their basic numbers, you can do the following: $3591 \div 21 = 171$.
Then $21^2 - 171 = 270 \div 3 = 90$.
Now find two numbers which sum is 21 and which product is 90. This will be 6 and 15.

Final solution: the basic numbers are 6, 15 and 17.

So it is crucial that the sum of the two remaining cubes is divisible by the sum of their basic numbers.

Summary:

1. $b^3 + c^3$ must be divisible by $b + c$ and destine the answer. $(3591 \div 21 = 171)$
2. Square $(b + c) = 21^2 = 441$

3. Subtract from this square the result of 1 (441 – 171 = 270)

4. Divide point 3 – point 2 by 3. The result must be divisible by 3. 270 ÷ 3 = 90

5. Find the two numbers which are the factors of point 4.

6. 21 ÷ 2 = 10,5

7. $10,5^2 = 110,25$

8. 110,25 – 90 = 20,25

9. $\sqrt{20,25} = \pm 4,5$

10. a = 10,5 + 4,5 = 15

11. b = 10,5 – 4,5 = 6

$a^3 + b^3 + c^3 = 140939$

Problem: $a^3 + b^3 + c^3 = 140939$ and $a + b + c = 83$

We have $a^3 + b^3 + c^3 = 140939 = 26 \pmod{27}$ The maximum possible number of one of the cubes is a^3, b^3 and c^3 is 52, as $52^3 = 140608$ and 53^3 is already larger than $a^3 + b^3 + c$, as it is 148877.

From table 3 we see that we should start with looking for a number which cube gives us a remainder 26 (mod 27).

We make a list of candidate numbers, which cubes are 26 (mod 27). They are : 44, 35, 26, 17 and 8.

First sieve: suppose $a = 44$, so $44^3 = 85184$. $140939 – 44^3$, $85184 = 55755$, and $b + c = a + b + c – a$. $83 – 44 = 39$. The maximum sum of the cubes of two numbers which add up to 39 is $38^3 + 1^3 = 54872 + 1 = 54873$. Too small! So 44 drops out.

Second sieve: $a = 35$, so $a^3 = 35^3 = 42875$.

Then we have $b^3 + c^3 = a^3 + b^3 + c^3 - a^3$; $140939 – 42875 = 98064$ and $b + c = a + b + c - a = 48$. b cannot be larger than 46, as $46^3 = 97336$. and If $b = 46$, $c = 2$, so $b^3 + c^3 = 97336 + 8 = 97344$. Too small again, therefore 35 drops out as well.

Third sieve: $a = 26$, and $26^3 = 17576$.

Then we have $b^3 + c^3 = a^3 + b^3 + c^3 - a^3$, $140939 – 17576 = 123363$. Now and $b + c = a + b + c - a = 83 - 26 = 57$. b cannot be greater than 49, therefore the maximum of $b^3 + c^3 = 49^3 + 8^3 = 118161$, which is too small. Therefore 26 drops out.

Fourth sieve $a = 17$, and $17^3 = 4913$.

Then we have $140939 – 4913 = 136026$. And $b + c = a + b + c$

- a now are 83 - 17 = 66. b cannot be greater than 51, as 51^3 = 132651. If b = 51 then c has to be 15! Now $51^3 + 15^3$ = 132651 + 3375 = 136026, and here is our solution.

Final answer 140939 = $17^3 + 51^3 + 15^3$.

To think about

The lowest possible sum if b + c = 66 is $33^3 + 33^3$= 71874.

The table hereunder illustrates the increase of the cubes with b + c = 66.

b	+	c							Increase
33	+	33	=	35937	+	35937	=	71874	= × 66
34	+	32	=	39304	+	32768	=	72072	3
35	+	31	=	42875	+	29791	=	72666	12
36	+	30	=	46656	+	27000	=	73656	27
37	+	29	=	50653	+	24389	=	75042	48
48	+	18	=	110592	+	5832	=	116424	675
49	+	17	=	117649	+	4913	=	122562	768
50	+	16	=	125000	+	4096	=	129096	867
51	+	15	=	132651	+	3375	=	136026	972

In the most right column we see the increase of the sum of $b^3 + c^3$. The numbers 3, 12 et cetera indicate the difference between 71874 and the following numbers, divided by 66, the sum of b + c. It will strike you that all the numbers represent 3 times a square, respectively 1^2, 2^2 , 3^2 et cetera . This column obeys the formula the difference between the sum of two cubed numbers with a constant sum, in this case b + c = 66, is three times the square of the difference between the “B”, 2. From 33 to 50 the difference is 17, and if b = 50 and c = 16, the difference between their cubes is $3 \times 17^2 = 867$ times 66.

In fact it is not necessary to make a walk along all the possibilities. We do not yet know if a if whether a= 17 is indeed the real choice, but it can be. We then calculate $b^3+c^3 = 136026$ and go further.

At the moment we are aware that the sum of $33^3 + 33^3$ is far below 136026, we do not need to examine all the numbers following on 33. Then it is more practical to take a cubed number a bit less than 136026. Take for example $50^3 = 125000$, add $16^3 = 4096$, then we have 129096. We now know we “are warm”. After that the next step $51^3 + 15^3$ is a full hit.

This way of working is identical in all cases where the sum of b + c represents an even number.

$a^3 + b^3 + c^3 = 359982$

Problem $a^3 + b^3 + c^3 = 359982$ and $a + b + c = 138$.

We have $a^3 + b^3 + c^3 = 359982 = 18 \pmod{27}$.

The maximum possible number of one of the cubes a^3, b^3 and c^3 is 71 as $71^3 = 357911$.

According to the table 2 the best approach for this question is to take a number which is divisible by 3, so that $a^3 = 0 \pmod{27}$.

We make a provisional list of some numbers < 71, and then get, counting down, respectively 69, 66, 63, 60, 57, 54, 51, 48. If this list appears to be too short, later on we continue.

Now we start.

$a = 69^3 = 328509$; 359982 – 328509 = 31.473. and $b + c = 69$. . Here we can stop immediately, as the lowest sum of two cubes, of which the basic number sum is 69, is $35^3 + 34^3$: 42875 + 39304 = 82179. This is so far over the mark, full stop.

$a = 66$. We get $a^3 = 66^3$ and $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 359982 – 287496 = 72486$. Now $b + c = 72$. The lowest possible sum of two cubes which' sum is 72 is $36^3 + 36^3 = 93312$. Again: here we can stop immediately. You'll see: if you look over the edge of your cupboard, a lot of unnecessary efforts can be economized!

$a = 63$. We get $a^3 = 250047$. $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 359982 – 250047 = 109935$. And $b + c = 75$. Lowest possible sum of two cubes which' sum is 75 is $38^3 + 37^3 = 105525$.

In the case if $b + c$ represents an odd number, it works according to the table hereunder.

38	+	37	=	54872	+	50653	=	105525	= ×225
39	+	36	=	59319	+	46656	=	105975	2
40	+	35	=	64000	+	42875	=	106875	6
41	+	34	=	68921	+	39304	=	108225	12
42	+	33	=	74088	+	35937	=	110025	20

Where do the numbers in the last column 2, 6, 12, 20 come from? We see 2, 6, 12, 20 and conclude: the difference is progressive and the following results can be predicted: 4, 6, 8 and the next one will be 10, then 12, 14 et cetera.

This differs from 33 + 33 as here the numbers differ and the sum is odd.

In the meantime we know that $a = 63$ is not the number we want and continue with $a = 60$.

$a = 60$, so $a^3 = 216000$. $B^3 + c^3 = 359982 - 216000 = 143982$ and $b + c = 78$. The lowest possible sum of two cubes which' sum is 78 is $39^3 + 39^3 = 118638$. Max $b = 52$, as $52^3 = 140608$. If $b = 51$, $c = 27$ and $b^3 + c^3$ is over the mark. $51^3 + 27^3 = 132651 + 19683 = 152334$, too much. One step down then: $50^3 + 28^3$. As this ends on $00 + 52 = 52$, this combination drops of. Conclusion: $a = 60$ is no good choice as well.

Another step down, $a = 57$.

We get $a = 57$, so $a^3 = 185193$, $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 359982 - 185193 = 174789$. Now $b + c = 138 - 57 = 81$. We calculate $54^3 + 27^3 = 19683 \times 9 = 177147$. This is too much. A number lower gives $53^3 + 28^3$. Here a rough estimation will do: $53^3 = \pm 149000$ and $28^3 = \pm 22000$, this is too low. Besides: 53^3 ends on 877 and 28^3 ends on 952, the sum is 829. We can finish our job with $a = 57$.

And $a = 54$? We would get $a^3 = 54^3 = 157464$. $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 359982 - 157464\ (54^3) = 202518$. If $a = 54$, then $b + c = 84$ and $b^3 + c^3 = 202518$. Again an estimation: $56^3 + 28^3 = 9 \times 21952\ (28^3) = 197568$. One number higher: $57^3 = 185193 + 27^3 = 19683 = 204876$. This is over the mark, 54 drops out.

$A = 51$. We would get $a^3 = 132651$. $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 359982 - 132651\ (51^3) = 227331$, and $b + c = 87$. We estimate $58^3 + 29^3$, which agrees with $9 \times 24389 = 219501$. We are too low. One step higher, this is more than only a try: 59^3 ends on 79; 28^3 ends on 52, therefore $59^3 + 28^3$ ends in 31. That sounds good!!

And indeed, this is a full hit as $59^3 + 28^3 = 227331$. Now we have found the final solution: $a + b + c = 138$ and $51^3 + 28^3 + 59^3 = 359982$.

It all seems to be a plunge in murky water. Not completely. A lot of useless work can be spared if one starts with a well-considered choice. For this table 2 is very helpful.

After having found the solutions of over a hundred of this type of questions, I thought: "Well, it's all clear to me, what can happen?". It is sometimes rather elaborate. E.g. the number 1191445 required eighteen sievings (!!), a lot of work, very challenging, very instructive, but it was all according to the standard path. And it is very instructive, because you learn to filter out the impossible candidates. And then came, purely by coincidence, the number 436510, which appeared to be a very unmanageable number.

$a^3 + b^3 + c^3 =$436510, the stumble stone

Problem $a^3 + b^3 + c^3 = 436510$ and $a + b + c = 154$.

We have $a^3 + b^3 + c^3 = 436510 = 1 \pmod{27}$. A +b+ c = 154.

The maximum possible number of one of the cubes a^3, b^3 and c^3 is 75, as $75^3 = 421875$. Following table 2 the attack on the number started with cubed numbers 1 (mod 27). These are the cubes of 73, 64, 55, 46, 37, 28, 19, 10, 1.

We start:

a = 73. We would get $a^3 = 389017$. $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 436510 - 389017(a^3) = 47493$. B + c then are 154 – 73 = 81. The lowest possible sum for $b^3 + c^3 = 41^3 + 40^3 = 132921$, which is far over the mark.

a = 64. We would get $a^3 = 262144$. $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 436510 – 262144\ (64^3) = 174366$. B + c = 90,the lowest minimum of $b^3 + c^3$ is $2 \times 45^3 = 182250$. Again: over the mark.

Now you are given some other numbers and the argumentation of their drop out:

a = 37. Where $a^3 = 50653$, $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 436510 – 50653\ (37^3) = 385857$. B + c = 117, the lowest minimal sum of $b^3 + c^3$ is $59^3 + 58^3 = 205379 + 195112 = 400491$, over the mark.

a = 28, we would get $a^3 = 21952$. $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 436510– 21952 = 414558$. B + c = 126, minimal sum of $b^3 + c^3 = 63^3 + 63^3 = 500094$.

We can stop here, as the lower numbers obviously lead to impossibilities: the sum of minimal sum of $b^3 + c^3$ = gets bigger and bigger.

We now conclude that one of the numbers, be it a, be it b or be it c, lies between 56 and 46, but is not necessarily one of these two numbers. We start with 55 and then count down, and will stop as soon as we see the number leads to a dead end.

a = 55, $a^3 = 166375$. . $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 436510 – 166375 = 270135= b^3 + c^3$. B + c = 99. Minimum sum of $b^3 + c^3 = 50^3 + 49^3 = 125000 + 117649 = 242649$. If $b^3 + c^3 = 60^3 + 39^3 = 275319$, too much. Alternative: $59^3 + 40^3 = 269379$. Too few. So a = 55 is not the right approach.

a = 54, where $a^3 = 157464$. . $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 436510 – 157464 = 279046$. B + c = 100. Minimum sum of $b^3 + c^3 = 50^3 + 50^3 = 250000$. The approach $b^3 + c^3 = 60^3 + 40^3 = 280000$ is too big, we do not need to calculate. The difference between 280000 – 250000 is 30000 = 100 × 300, the minimum difference between $b^3 + c^3 = 50^3 + 50^3$ and $b^3 + c^3 = 51^3 + 49^3 = 250300$. The increase of the combinations $b^3 + c^3$ in this case is $x^2 \times 300$. Stepping from 50 to 60 we need 10 numbers, then the increase of $b^3 + c^3$ if $b + c = 100 = 300$ per number.

After this the conclusion can only be: also 54 drops out.

Well, next approach: $a = 53$ and a^3 is 148877. What we get now is $b^3 + c^3 = a^3+b^3+c^3-a^3 = 436510 – 148877 = 287633$.

As we know that now $b = 60$ and $60^3 = 216000$ and c^3 (41^3) = 68921, and finally $b^3 + c^3 = 284921$. Next approach $b = 61$ and $c = 40$, $= b^3 + c^3 = 226981 + 64000 = 290921$. Here we conclude that $a = 53$ does not result in a correct solution. Up to the next number: $a= 52$.

$a = 52$, $52^3 = 140608$. $b^3 + c^3 = a^3 + b^3 + c^3 - a^3 = 436510 – 140608 = 295902$. $B + c= 102$, the minimal sum of $b^3 + c^3 = 51^3 + 51^3= 265302$. The difference with the combination $52^3 + 50^3 = 140608 + 125000 = 265608$ is 306. We do not need to make all the steps, and make an approach: 295902 – 265302 = 30600.

Well, that's a surprise: 30600 ÷ 306 = exactly 100 times. As 100 is the square of ten, we now have found that the correct combination for $b^3 + c^3$ is $61^3 + 41^3$: 226981+ 68921 = 295902.

Another very important thing is looking for the final figures: we go back to $b + c = 102$. If we take e.g. $52^3 + 50^3$ the result ends on 608, more precisely 265608. And $53^3 + 49^3$ ends on 77+ 49 Ξ 26, therefore neither this combination is what we are looking for. This is a matter of esteeming and thinking: for obtaining 295902 we can esteem: $60^3 + 40^3 = 280.000$ so we immediately see we have to take bigger numbers, and $61^3 + 41^3$ ends on 902. As there is no combination possible, we calculate this exactly and see it was the correct choice.

The final solution for this question is a = 52, b = 61 and c = 41.

On the internet we can almost daily read something about a world record. This is in fact a convicting proof of ones skills. If one should ask me if this kind of questions lends itself for a world record attempt, I can only say: NO.

Why that? Well, you see it here above: for finding the numbers we want, there is no straight ahead method. It is a matter of sieving and thinking, and let's not forget: a profound knowledge of the cubes up to hundred. Besides it is a matter of combination of the last figures of a cube, to see if the estimation is a good one or if we have to continue.

The deep joy is to be found in the satisfaction of having found the answer on a kind of question for which until now no form exists for finding the answer in one straight ahead move.

And there is more: the well- known English mental calculation prodigy George Lane once wrote that all the kinds of arithmetic operations on one or another way are interlinked. I can only confirm. Also this kind of questions, although they may seem to be absurd, really enrich the feeling with numbers, it is an exercise for training in the knowledge of the cubes, addition, subtraction, division and logical thinking.

And the most: the satisfaction of finding the solution!!