It can still be made tougher

In the book “The great Mental Calculators” we find also a question like the following ones: given the sum of three numbers, given the sums of their squares and their cubes, and given the product of the sum of their squares times their cubes”: a+b+c = 65; $(a^2+b^2+c^2)*(a^3+b^3+c^3)$ = 70.405.013 – find the three numbers. The answer is given, the solution is not.

Dr. B. de Weger suggested that after factorisation of the question number there were a possibility to find the solution.

During the “Rekenwonderweekend 2017” where my guests were Andy Robertshaw, Jasper Visser and Kenneth Wilshire this type of question was the leading issue. In this article I’ll work out a number of questions and give further information. For the good order: the columns a, b and c and the sum of the squares and cubes were hidden. The column of the factors was made by ourselves, as in our feeling without that it is impossible to find the answer. What is given is printed in green.

a	b	c	a+b+c	$a^2+b^2+c^2$	$a^3+b^3+c^3$	product $(a^2+b^2+c^2)*(a^3+b^3+c^3)$	factors
14	17	19	50	846	14516	12280536	2*2*2*3*3*19*47*191

Minimum sum of the squares of $17^2+17^2+16^2 = 834$. Maximum sum $48^2+1^2+1^2= 2306$.

The sum of the three squares gives us this:

- None of the numbers is divisible by 3, as squares of numbers not 0 mod 3 are always 1 mod 3 and therefore the sum of the squares is 0 mod 3.
- As the sum of the squares is even, there is one basic number an even one, the two other ones are odd
- The sum of the cubes is 8 mod 9 which means that there are two cubes 8 mod 9 and one cube is 1 mod 9.

What we do is making a choice out of the factors, destine their product and find the squares which compose the product.

As an estimation we take 2*3*3*47 = 846, as we have to realise that we also have to consider the sum of the cubes and the value of the product.

In the table we make a selection of the numbers which squares have the sum 846.

a	b	c	Sum	24	/		
29	2	1	32	23	14	11	48
28	/			22	19	1	42
27	9	6	42	21	18	9	48

26	13	1	40	20	/		
26	11	7	44	19	17	14	50
25	11	10	46	18	Known		
25	14	5	44	17	Known		

/ Means there is no solution for the sum.

We find that the only combination with the sum 846 are the squares of 14, 17 and 19.

To check we calculate the sum of the cubes of the numbers we found.

19^3	6859
17^3	4913
14^3	2744
	14516

We now may conclude our solution is correct.

We cannot take 2*3*3*19 = 228, this is below the minimum sum of the squares which is $17^2+17^2+16^2 = 834$ and $48^2 + 1^2 + 1^2 = 2306$ is the maximum theoretic sum of the three squares. .

Another number.

a	b	c	a+b+c	$a^2+b^2+c^2$	$a^3+b^3+c^3$	product $(a^2+b^2+c^2)*(a^3+b^3+c^3)$	factors
5	29	32	66	1890	57282	108262980	2*2*3*3*3*3*5*7*9547

Reasoning: the product contains 2^3, so there is one even number and two odd ones. For: if there were three even numbers, the sum of the squares would have at least 2^2 and the sum of the cubes would at least have 2^3. Minimum sum of the squares $3\times 22^2= 1452$, maximum sum of them $64^2+1^2+1^2 = 4098$.

Again: there is no standard criterion for which of the factors we take, but we can for finding a sum of three squares simply ignore 9547. We have to take more than 1452, reasonable is 1890 as the product of 2*3*3*3*5*7 = 1890.

The table:

43	5	3	51		37	20	11	68	
42	/				36	/			
41	/				35	/			
40	13	11	64		34	/			
40	17	1	58		33	24	15	72	
39	15	12	66	candidate	32	29	5	66	candidate
38	/				31	23	20	74	

We have two candidates. 39, 15 and 12 are all divisible by three which implicates that the sum of the squares will be divisible by at least 27 as this is 3 × 9 and the sum of the cubes will be divisible by 27, so the product will have 3^6. As we have in the factorisation 3^4 this possibility can be ignored. What remains are the numbers 32, 29 and 5.

The table gives us

32	32768
29	24389
5	125
	57282

This result is divisible by 3, as to be expected, in combination with the 3^3 from the sum of the squares and finally we multiply 1890 × 57282 = 108262980, so this is the correct solution.

Our last example:

a	b	c	a+b+c	$a^2+b^2+c^2$	$a^3+b^3+c^3$	product $(a^2+b^2+c^2)*(a^3+b^3+c^3)$	factors
47	31	22	100	3654	144262	527133348	2*2*3*3*7*17*29*4243

Minimum sum of the squares $33^2+33^2+34^2 = 3334$, maximum sum of them $98^2+1^2+1^2= 9606$.

The product contains 2^2, from which we conclude that there are two odd numbers and one even one. The product contains also 3^2. This can come from the squares in this modulo 9 combination: 1+1+7 or 7+7+4. This means that the sum of the cubes cannot be 0, 3 or 6 mod 9.

How shall we start with the sum of squares? 2*4243= 8486 is far over the mark, as we later on have to take the same numbers for the sum of the cubes. It is a rough approach, let's take the rounded fifth root of the question number, ± 55 and square that to get an idea. So somewhere in the 3.000.

We gamble 2*3*17*29= 2958 and make a table:

54	/				31	34	29	94
53	10	7	70		30	/		
52	/				29	46	1	
51	/				29	34	31	94
50	17	13	80		28	/		
49	19	14	82		27	/		
48	/				26	/		
47	/				25	43	22	90
46	/				24	/		
45	/				23	/		
44	/				22	43	25	90
43	25	22	90		21	/		
42	/				20	/		
41	34	11	86		19	49	14	82
40	/				18	/		
39	/				17	50	13	80
38	33	5	76		16	/		
37	/				15	/		
36	/				13	50	17	80
35	38	17	90		12	/		
34	31	29	94		11	41	34	86
34	41	11	86					
33	/							
32	/							

We conclude that there is no solution for this combination. Therefore we take 2*3*3*7*29 = 3654 and make a new table.

59	13	2	74		47	34	17	98
58	17	1	76		47	38	1	85
58	13	11	82		47	31	22	100
57	18	6	81		46	37	13	96
56	/				45	30	27	102
55	25	2	82		44	/		
55	23	10	88		43	38	19	100
54	27	3	84		42	/		
53	29	2	84		41	38	23	102
53	26	13	92		40	/		
52	/				39	/		
51	27	18	96		38	37	29	104
50	25	23	98		38	47	1	86
49	/				37	46	13	96
48	/				36	/		
					35	/		

We have two candidate solutions, which we compare:

B.N.	Cubes		B.N.	Cubes
47	103823		43	79507
31	29791		38	54872
22	10648		23	12167
sum	144262		sum	146546

The multiplication check: a number xx54 × xx62 results in a number ending on xx48 and xx46 × xx62 results in a number xx52. So the combination 47, 31 and 22 is the correct solution.

After having consulted some mathematicians for me remain these conclusions:

- This kind of questions can only be solved after firstly having factorised the product
- It is inevitable to use a factorisation program
- For finding a number for the sum of three squares there is only the “brute force attack”, as the mathematicians mention it.

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47	/				25	43	22	90
46	/				24	/		
45	/				23	/		
44	/				22	43	25	90
43	25	22	90		21	/		
42	/				20	/		
41	34	11	86		19	49	14	82
40	/				18	/		
39	/				17	50	13	80
38	33	5	76		16	/		
37	/				15	/		
36	/				13	50	17	80
35	38	17	90		12	/		
34	31	29	94		11	41	34	86
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After having consulted some mathematicians for me remain these conclusions:

- This kind of questions can only be solved after firstly having factorised the product
- It is inevitable to use a factorisation program
- For finding a number for the sum of three squares there is only the “brute force attack”, as the mathematicians mention it. There seems not to be a method to find the sum of the three squares by means of a formula.

Alphen aan den Rijn, July 2017, Willem Bouman