Diophantine Equations and Modulo 11.

Those who were present during the Mental Calculation World Cup will remember that from Andreas Berger and Andy Robertshaw came the question “Is there always one solution possible or are there more?”

Shortly after that Andy made a document in which appeared that there are quite a lot of numbers with at least 2 solutions for the questions a + b + c = x and $a^3 + b^3 + c^3$ = y. Another question, from Ralf Laue “You did this with modulo 27, isn’t it possible to find the solution with modulo 11?”.

Yes, this can also be done. My friend Dr. Benne de Weger engaged the Mathematica program. It calculated for all the moduli 11 0-10 the big sum, the addition of three cubed numbers and the small sum, the modulo 11 of the basic numbers. I do not give the complete table, we take 3 numbers with 2 solutions and work this out.

810.123 – $a^3 + b^3 + c^3$ - the big sum – is 6 (11). And a + b + c – the small sum - 147 = 4 (11).

The table gives this:

{**6**, {0, 0, 8}, {0, 1, 3}, {0, 2, 4}, {0, 5, 7}, {0, 6, 10}, {0, 9, 9}, {1, 1, 5}, {1, 2, 2}, {1, 4, 6}, {1, 7, 9}, {1, 8, 10}, {2, 3, 5}, {2, 6, 7}, {2, 8, 9}, {2, 10, 10}, {3, 3, 6}, {3, 4, 9}, {3, 7, 10}, {3, 8, 8}, {4, 4, 10}, {4, 5, 5}, {4, 7, 8}, {5, 6, 8}, {5, 9, 10}, {6, 6, 9}, {7, 7, 7}},

The bold printed number 6 is the mod 11 of 810.123, the mod 11 of the small sum is 4. Now we look for the numbers between the accolades which sum is 4 mod 11. We see {0, 1, 3}. We cube these numbers and find 0 + 1 + 27 = 28 = 6 (11). And 2, 6, 7. Check, the sum of their cubes is 8 + 216 + 343 = 567, which is – of course- also 6 (11).

Next: cube root of 810123 = 93 + and we take the numbers below 93 which are 0 (11).

Big sum	B.N.	Cube	BS – Cube	147 - BN	divided	
810123	88	681472	128651	59		Remainder
	77	456533	353590	70		Remainder
	66	287496	522627	81		Remainder
	55	166375	643375	92		Remainder
	44	85184	724939	103		Remainder
	33	35937	774186	114		Remainder
	22	10648	799475	125		Remainder
	11	1331	808792	136	5947	Integer

Now we subtract the cubes of the 0 mod 11 numbers from the big sum, 4^{th} column and we subtract the 0 mod 11 numbers from the small sum, column 5. Then we divide the column 4 numbers by the column 5 numbers and find there is one division without remainder: 808792 ÷ 136 = 5947. Our next step is to find the numbers which’ cubes added result in 808792 AND – not to forget- which sum is 6 mod 11. Over more the sum of the basic numbers has to be 147 – 11 = 136.

Given is that one of these numbers is 1 mod 11 and the other one is 3 mod 11. Their maximum is 93.

3 mod 11	Cubed	1 mod 11	Cubed
91	753571	89	704969
80	512000	78	474552
69	328509	67	300763
58	195112	56	175616
47	103823	45	91125
36	46656	34	39304
25	15625	23	12167
14	2744	12	1728
3	27	1	1

As the sum is 136, the half of it is 68. Next: $68^3 = 314432 \times 2 = 628864$. Then $808792 - 628864 = 179928 \div 136 = 1323 \div 3 = 441$ and $\sqrt{441} = \pm 21$. So $b = 68 + 21 = 89$ and $c = 68 - 21 = 47$.
Now we try to find the solution for the option 2, 6, 7. And take, randomly the middle one: 6.

Big sum	B.N.	Cube	BS - Cube	147- BN	Divided	
810123	83	571787	238336	64	3724	Integer
	72	373248	436875	75		Remainder
	61	226981	583142	86		Remainder
	50	125000	685123	97		Remainder
	39	59319	750804	108		Remainder
	28	21952	788171	119		Remainder
	17	4913	806110	130		Remainder
	6	216	809907	141		Remainder

Next question: $b + c = 64$, one of these numbers is 7 (11) and the other one is 2 (11), and $b^3 + c^3 = 238336$. The biggest one is 62 as $62^3 = 238328$. As 238336 is divisible by 16, this implies that the basic numbers must be even. Then the only possible solution is 62 and 2, so the final solution for a, b and c is 83, 62 and 2. This is reasoning, next we calculate.

We take $64^2 = 4096$ and subtract 3724 and get 372. Then $372 \div 3 = 124$, now look for two numbers sum 64 and product 124. Here we quickly find the numbers 62 and 2, by which our final solution gives us the numbers 83, 62 and 2.

Next number big sum 409473, 9 (11), cube root of it 74+; small sum 129 = 8 (11). The table for 9 (11) and sum 8 (11):

{**9**, {0, 0, 4}, {0, 1, 2}, {0, 3, 5}, {0, 6, 7}, {0, 8, 9}, {0, 10, 10}, {1, 1, 6}, {1, 3, 9}, {1, 4, 10}, {1, 5, 5}, {1, 7, 8}, {2, 2, 5}, {2, 3, 6}, {2, 4, 9}, {2, 7, 10}, {2, 8, 8}, {3, 3, 10}, {3, 4, 8}, {3, 7, 7}, {4, 4, 7}, {4, 5, 6}, {5, 7, 9}, {5, 8, 10}, {6, 6, 8}, {6, 9, 10}, {9, 9, 9}},

We find 0, 3, 5; other option 2, 7, 10. We take 5 mod 11 and get

Big sum	B.N.	Cube	BS - Cube	129 - BN	divided	
409473						
	71	357911	51562	58	889	Integer
	60	216000	193473	69		Remainder
	49	117649	291824	80		Remainder
	38	54872	354601	91		Remainder
	27	19683	389790	102		Remainder
	16	4096	405377	113		Remainder
	5	125	409348	124		Remainder

The sum of b + c will be 129 – 71 = 58 and modulo 11: 8 (11) – 5 (11) = 3 (11); the sum of $b^3 + c^3 = 51562$.

Now we work with the table here under. We realise that the sum of $b^3 + c^3$ does not exceed 51162, means that we ca**n** ignore the numbers greater than 36. The only combination in both columns which fits is 33 + 25. Check: $33^3 + 25^3 = 35937 + 15625 = 51562$.

3 mod 11	Cubed	0 mod 11	Cubed
36	46656	33	35937
25	15625	22	10648
14	2744	11	1331
3	27		

Another method to find b and c is the following. We know $b^3 + c^3 = 51562$ and b + c = 58. We divide 58 ÷ 2 = 29 and take $2 \times 29^3 = 48778$. Next 51562 – 48778 = 2784 and 2784 ÷ 58 = 48. Then 48 ÷ 3 = 16 and sqrt 16 = ± 4. Finally we calculate 29 ± 4 and get 33 and 25, with which we now know b and c.

The final solution is: a, b and c are 71, 33, 25.

About the tables

{**6**, {0, 0, 8}, {0, 1, 3}, {0, 2, 4}, {0, 5, 7}, {0, 6, 10}, {0, 9, 9}, {1, 1, 5}, {1, 2, 2}, {1, 4, 6}, {1, 7, 9}, {1, 8, 10}, {2, 3, 5}, {2, 6, 7}, {2, 8, 9}, {2, 10, 10}, {3, 3, 6}, {3, 4, 9}, {3, 7, 10}, {3, 8, 8}, {4, 4, 10}, {4, 5, 5}, {4, 7, 8}, {5, 6, 8}, {5, 9, 10}, {6, 6, 9}, {7, 7, 7}},

Here we see a wide variety of the possibilities with mod 11, it is very interesting to work such a table out. So for all the possibilities is valid that the sum of the cubes of the given numbers, the big sum, is 6 mod 11. And the small sum can be any number between 0 and 10 mod 11, as is shown in the following table.

Basic numbers	Small sum	Basic numbers	Small sum
1, 4, 6	0	0, 9, 9	7
2, 10, 10	0	1, 1, 5	7
0, 5, 7	1	4, 4, 10	7
3, 3, 6	1	0, 0, 8	8
5, 9, 10	2	1, 8 10	8
4, 5, 5	3	2, 8, 9	8
0, 1, 3	4	3, 8, 8	8
2, 6, 7	4	5, 6, 8	8
1, 2, 2	5	4, 7, 8	8
3, 4, 9	5	3, 7, 10	9
0, 6, 10	5	2, 3, 5	10
1, 7, 9	6	6, 6, 9	10
0, 2, 4	6	7, 7, 7	10

It is on purpose that I did not print the complete table, that will take too much paper. It concerns 10 moduli 11 with each of them 26 possibilities. And those who want it, please mail me and you get it the complete table. It is a challenge to create such a table, but it is really time consuming. For finding the numbers for big sum 9 mod 11 and small sum 2 mod 11 it took me five minutes to find the combinations {1, 3, 9} and {0, 6, 7}.

Have a look at the numbers small sum 8 mod 11. Here we see 5 combinations and in each of everyone there are two numbers with the sum 0 mod 11, + 8: 0 + 0; 1 + 10; 2 + 9; 3 + 8; 4 + 7; 5 + 6.

Finally we work out the number big sum 1271791, 4 mod 11 and small sum 175, 10 mod 11.

The table gives as possibilities { 4, 2, 4} and {0, 1, 9}. We start with 4 mod 11 and then get:

Big sum	B.N.	Cube	BS - Cube	175 - BN	divided	
1271791	103	1092727	179064	72	2487	Integer
	92	778688	493103	83		Remainder
	81	531441	740350	94		Remainder
	70	343000	928791	105		Remainder
	59	205379	1066412	116		Remainder
	48	110592	1161199	127		Remainder
	37	50653	1221138	138		Remainder
	26	17576	1254215	149		Remainder
	15	3375	1268416	160		Remainder
	4	64	1271727	171	7437	Integer

Although the division by 72 is an integer one, there is no solution: the smallest sum of two numbers squared with sum 72 is $36^2 \times 2 = 2592$.

Next: $171^2 = 29241$ and 29241 – 7437 = 21804, and 21804 ÷ 3 = 7268. So now we look for two numbers which sum is 171 and which product is 7268. 171 ÷ 2 = 85,5 and $85{,}5^2$ = 7310,25. 7310,25 – 7268 = 42,25 and $\sqrt{42{,}25} = \pm 6{,}5$. Then 85,5 + 6,5 = 92, this is the number b and 85,5 – 6,5 = 79, this is c. Final solution 1271791 is the sum of the cubes of 4, 92 and 79.

One More Example

Hereunder you'll find the work out of the question a + b + c = 108 = 9 mod 11 the small sum and $a^3 + b^3 + c^3 = 226962 = 10 \bmod 11$, the big sum . The table gives as possibilities a, b and c are 0, 2, 7, sum = 9 mod 11 ; 1, 3 and 5, sum 9 mod 11; 4, 8, 8 sum 9 mod 11.

We start with 0 mod 11, the biggest number 0 mod 11 is 55.

Big sum	0 mod 11	0 mod 11^3	QN – 0 mod 11 ^3	108 – 0 mod 11	Int. div
226962	55^3	166375	60587	53	No
	44^3	85184	1417784	64	No
	33^3	35937	191025	75	2547
	22^3	10648	212314	86	No
	11^3	1331	225631	97	No

We look for an integer division. The result is to be found in the column Int. div. Now we take 108 – 33 = 75 and square this :5625. Next 56215 – 2547 = 3078 and 3078 ÷ 3 = 1026. Finally we look for two numbers which sum is 75 and which product is 1026. We find 18 and 57. Final solution. The big sum 226962 is the sum of the cubes 33, 18 and 57.

It works also if we take numbers 2 mod 11. As follows:

Big sum	2 mod 11	2 mod 11^3	QN – 2 mod 11 ^3	108 – 2 mod 11	Int. div
226962	57	185193	41769	51	819
	46	97336	129626	62	no
	35	42875	184087	73	no
	24	13824	213138	84	no
	13	2197	224765	95	no
	2	8	226954	106	no

Now $51^2 = 2601 - 819 = 1782$ and 1782 ÷ 3 = 594. Now we look for two numbers which sum is 52 and which product is 594. These numbers are 18 and 33 so the final solution is 57, 18 and 33.

And if we take 7 we see this

Big sum	7 mod 11	7 mod 11^3	QN – 7 mod 11 ^3	108 – 7 mod 11	Int. div
226962	51	132651	94311	57	no
	40	64000	162962	68	no
	29	24389	202573	79	no
	18	5832	221130	90	2457
	7	343	226619	101	no

Now $90^2 = 8100$ and 8100 – 2457 = 5643. Next 5643 ÷ 3 = 1881. Which numbers have sum 90 and product 1881? Well these are 33 and 57 and \again we find the numbers 57, 18 and 33.

Next search we examine the possibilities 1, 3 and 5, and start with 1 mod 11.

Big sum	1 mod 11	1 mod 11^3	QN – 1 mod 11 ^3	108 – 1 mod 11	Int. div
226962	56	175616	51346	52	No
	45	91125	135837	63	No
	34	39304	187658	74	No
	23	12167	214795	85	2527
	12	1728	225234	96	No
	1	1	226961	107	N0

Now $85^2 = 7225$ and 7225 – 2527 = 4698 and 4698 ÷ 3 = 1566. Next we look for 2 numbers which sum is 85 and which product is 15667. These numbers are 27 and 58, so our final solution is 226962 is the sum of the cubes 23, 27 and 58.

Now we see confirmed the idea of Andreas and Andy: there are 2 solution for the number 226962.

To demonstrate that it all works perfectly we take 3 mod 11.

Big sum	3 mod 11	3 mod 11^3	QN – 3 mod 11 ^3	108 – 3 mod 11	Int. div
226962	58	195112	31850	50	637
	47	103823	123139	61	No
	36	46656	180306	72	No
	25	15625	211337	83	No
	14	2744	224218	94	No
	3	27	226935	105	N0

We take $50^2 = 2500$ and subtract 637 and get 1863. 1863 ÷ 3 = 621 and look for the numbers with sum 50 and product 621. They are 23 and 27 and now have the solution: 226962 is the sum of the cubes of 58, 23 and 27.

Finally we take the numbers 5 mod 11 and get:

Big sum	5 mod 11	5 mod 11^3	QN – 5 mod 11 ^3	108 – 5 mod 11	Int. div
226962	60	216000	10962	48	No
	49	117649	109313	59	No
	38	54872	172090	70	No
	27	19683	207279	81	2559
	16	4096	222866	92	No
	5	125	226137	103	No

Now we do $81^2 = 6561$ – 2559 = 4002. 4002 ÷ 3 = 1334. Which numbers have sum 81 and product 1334? They are 23 and 58, so the final answer for 226962 is the cubes of 27, 23 and 58.

How many?

We now know that there are many times two solutions for a, b and c. In the “Andy“- file we the number 538237 with four different possibilities. The file reaches up to 200, in this range 4 different combinations is the maximum.

There is the combination 0, 2, 10 and working this out I found 33, 79, 21 and 77, 13, 43. Two different solutions with the same modulus 11! So every possibility has to be worked out completely.

In the table here above we can easily see that as always there are 26 combinations, there is no “honest” subdivision: there are sums with only 1 combination, also 2 and we find even 5 combinations with the sum 8.

Over all, it was an instructive and interesting challenge to work this out. Thanks to them who did the suggestion!!

Best regards,

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